S.-T. Yau College Student Mathematics Contests 2018

Oral Exams in Geometry and Topology

Individual (4 problems)

1. Is the surface of genus 3 a covering space of the surface of genus 2. Is the surface of genus 2 a covering space of the surface of genus 3. If so, then exhibit an example of such a covering.

2. Consider a smooth map $f : S^{2n-1} \to S^n$ with $n \geq 2$. Let ν be a volume form on S^n with volume 1.

a). Show that $f^*\nu$ is exact.

b). Write $f^*\nu = d\alpha$. Show that the integral

$$\int_{S^{2n-1}} \alpha \wedge f^*\nu$$

is independent of the choice of α.

c). Show that the integral above is actually an invariant of the homotopy class of f.

d). Show that the integral is 0 if n is odd.

e). Calculate this integral if f is the Hopf map $(z_0, z_1) \to [z_0, z_1]$.

3. (D. Hilbert) There does not exist a complete regular surface in $\mathbb{R}^3$ whose Gaussian curvature is a negative constant K_0.

4. Let $\overline{CP^2}$ denote the complex projective plane with the opposite orientation.

a). Show that $S^2 \times S^2$ and $CP^2 \# \overline{CP^2}$ have the same cohomology group but different cohomology ring.

b). Show that $(S^2 \times S^2) \# \overline{CP^2}$ and $CP^2 \# 2\overline{CP^2}$ have the same cohomology.